The Yule Walker Equations for the AR Coefficients

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If you assume a given zero-mean discrete timeseries $\{x_i\}_1^N$ is an AR process, you will naturally want to estimate the appropriate order p of the AR(p),

$$x_{i+1} = \phi_1 x_i + \phi_2 x_{i-1} + \cdots + \phi_p x_{i-p+1} + \xi_{i+1} \tag{1}$$

and the corresponding coefficients $\{\phi_j\}$. There are (at least) 2 methods, and those are described in this section.

1 Direct Inversion

The first possibility is to form a set of direct inversions,

1.1 $p = 1$

With

$$x_{i+1} = \phi_1 x_i + \xi_{i+1},$$

one can form the over-determined system

$$\underbrace{\begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_N \end{pmatrix}}_{\mathbf{b}} = \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{pmatrix}}_{\mathbf{A}} \phi_1$$

which can be readily solve using the usual least-squares estimator

$$\hat{\phi}_1 = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b} = \frac{\sum_{i=1}^{N-1} x_i x_{i+1}}{\sum_{i=1}^{N-1} x_i^2} = \frac{c_1}{c_o} = r_1$$

where c_i and r_i are the ith autocovariance and autocorrelation coefficients, respectively.

1.2 $p = 2$

With

$$x_{i+1} = \phi_1 x_i + \phi_2 x_{i-1} + \xi_{i+1},$$

start by forming the over-determined system

$$\underbrace{\begin{pmatrix} x_3 \\ x_4 \\ \vdots \\ x_N \end{pmatrix}}_{\mathbf{b}} = \underbrace{\begin{pmatrix} x_2 & x_1 \\ x_3 & x_2 \\ \vdots & \vdots \\ x_{N-1} & x_{N-2} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}}_{\mathbf{\Phi}}.$$

Unlike the previous $p = 1$ case, trying to express the solution

$$\hat{\mathbf{\Phi}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$$

analytically is not trivial. We start with

$$\left(\mathbf{A}^T\mathbf{A}\right)^{-1} = \left[\begin{pmatrix} x_2 & x_3 & \cdots & x_{N-1} \\ x_1 & x_2 & \cdots & x_{N-2} \end{pmatrix}\begin{pmatrix} x_2 & x_1 \\ x_3 & x_2 \\ x_{N-1} & x_{N-2} \end{pmatrix}\right]^{-1}$$

$$= \begin{pmatrix} \sum_{i=2}^{N-1} x_i^2 & \sum_{i=2}^{N-1} x_i x_{i-1} \\ \sum_{i=2}^{N-1} x_i x_{i-1} & \sum_{i=1}^{N-2} x_i^2 \end{pmatrix}^{-1}$$

$$= \frac{1}{\sum_{i=2}^{N-1} x_i^2 \sum_{i=1}^{N-2} x_i^2 - \sum_{i=2}^{N-1} x_i x_{i-1} \sum_{i=2}^{N-1} x_i x_{i-1}} \begin{pmatrix} \sum_{i=1}^{N-2} x_i^2 & -\sum_{i=2}^{N-1} x_i x_{i-1} \\ -\sum_{i=2}^{N-1} x_i x_{i-1} & \sum_{i=2}^{N-1} x_i^2 \end{pmatrix}.$$

Next, let's use the fact that the timeseries is stationary, so that autocovariance elements are a function of the lag only, not the exact time limits. In this case,

$$\left(\mathbf{A}^T\mathbf{A}\right)^{-1} = \frac{1}{c_o^2 - c_1^2}\begin{pmatrix} c_o & -c_1 \\ -c_1 & c_o \end{pmatrix},$$

$$\left(\mathbf{A}^T\mathbf{A}\right)^{-1} = \frac{1}{c_o^2(1 - r_1^2)}\begin{pmatrix} c_o & -c_1 \\ -c_1 & c_o \end{pmatrix},$$

$$\left(\mathbf{A}^T\mathbf{A}\right)^{-1} = \frac{1}{c_o(1 - r_1^2)}\begin{pmatrix} r_o & -r_1 \\ -r_1 & r_o \end{pmatrix}.$$

Similarly,

$$\mathbf{A}^T\mathbf{b} = \begin{pmatrix} x_2 & x_3 & \cdots & x_{N-1} \\ x_1 & x_2 & \cdots & x_{N-2} \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} \sum_{i=3}^{N} x_i x_{i-1} \\ \\ \sum_{i=3}^{N} x_i x_{i-2}, \end{pmatrix}$$

which, exploiting again the stationarity of the timeseries, becomes

$$\mathbf{A}^T\mathbf{b} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Combining the 2 expressions, we have

$$\begin{aligned} \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b} &= \frac{1}{c_o(1-r_1^2)} \begin{pmatrix} r_o & -r_1 \\ -r_1 & r_o \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \frac{1}{1-r_1^2} \begin{pmatrix} 1 & -r_1 \\ -r_1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \end{aligned}$$

Breaking this into individual components, we get

$$\hat{\phi}_1 = \frac{r_1\left(1-r_2\right)}{1-r_1^2}$$

and

$$\hat{\phi}_2 = \frac{r_2 - r_1^2}{1-r_1^2}$$

Of course it is possible to continue to explore $p \geq 3$ cases in this fashion. However, the algebra, while not fundamentally different from the $p = 2$ case, quickly becomes quite nightmarish. For example, for $p = 3$,

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} c_o & c_1 & c_2 \\ c_1 & c_o & c_1 \\ c_2 & c_1 & c_o \end{pmatrix},$$

whose determinant, required for the inversion, is the cumbersome-looking

$$\det\left(\mathbf{A}^T\mathbf{A}\right) = c_o\left(c_o^2 - 2c_1^2 + 2\frac{c_1^2 c_2}{c_o} - c_2^2\right) = c_o\left[c_o^2 + 2c_1^2\left(r_2 - 1\right) - c_2^2\right],$$

which, on pre-multiplying by the remainder matrix, yields very long expressions.

Fortunately, there is a better, easier way to obtain the AR coefficient for the arbitrary p, the Yule-Walker Equations.

2 The Yule-Walker Equations

Consider the general AR(p)

$$x_{i+1} = \phi_1 x_i + \phi_2 x_{i-1} + \cdots + \phi_p x_{i-p+1} + \xi_{i+1}.$$

2.1 Lag 1

- multiply both sides of the model by x_i,

$$x_i x_{i+1} = \sum_{j=1}^{p} \left(\phi_j x_i x_{i-j+1}\right) + x_i \xi_{i+1},$$

  where i and j are the time and term indices, respectively,

- take expectance,

$$\langle x_i x_{i+1} \rangle = \sum_{j=1}^{p} \left(\phi_j \langle x_i x_{i-j+1} \rangle\right) + \langle x_i \xi_{i+1} \rangle$$

  where the $\{\phi_j\}$s are kept outside the expectance operator because they are deterministic, rather than statistical, quantities.

- note that $\langle x_i \xi_{i+1} \rangle = 0$ because the shock (or random perturbation) ξ of the current time is unrelated to–and thus uncorrelated with–previous values of the process,

$$\langle x_i x_{i+1} \rangle = \sum_{j=1}^{p} \left(\phi_j \langle x_i x_{i-j+1} \rangle\right)$$

- divide through by $(N-1)$, and use the evenness of the autocovariance, $c_{-l} = c_l$,

$$c_1 = \sum_{j=1}^{p} \phi_j c_{j-1}$$

- divide through by c_o,

$$r_1 = \sum_{j=1}^{p} \phi_j r_{j-1}.$$

2.2 Lag 2

- multiply by x_{i-1},

$$x_{i-1}x_{i+1} = \sum_{j=1}^{p} \left(\phi_j x_{i-1} x_{i-j+1}\right) + x_{i-1}\xi_{i+1},$$

- take expectance,

$$\langle x_{i-1}x_{i+1}\rangle = \sum_{j=1}^{p} \left(\phi_j \langle x_{i-1} x_{i-j+1}\rangle\right) + \langle x_{i-1}\xi_{i+1}\rangle$$

- eliminate the zero correlation forcing term

$$\langle x_{i-1}x_{i+1}\rangle = \sum_{j=1}^{p} \left(\phi_j \langle x_{i-1} x_{i-j+1}\rangle\right)$$

- divide through by $(N-1)$, and use $c_{-l} = c_l$,

$$c_2 = \sum_{j=1}^{p} \phi_j c_{j-2}$$

- divide through by c_o,

$$r_2 = \sum_{j=1}^{p} \phi_j r_{j-2}.$$

2.3 Lag k

- multiply by x_{i-k-1},

$$x_{i-k+1}x_{i+1} = \sum_{j=1}^{p} \left(\phi_j x_{i-k+1} x_{i-j+1}\right) + x_{i-k+1}\xi_{i+1},$$

- take expectance,

$$\langle x_{i-k+1}x_{i+1}\rangle = \sum_{j=1}^{p} \left(\phi_j \langle x_{i-k+1} x_{i-j+1}\rangle\right) + \langle x_{i-k+1}\xi_{i+1}\rangle$$

- eliminate the zero correlation forcing term

$$\langle x_{i-k+1}x_{i+1}\rangle = \sum_{j=1}^{p} \left(\phi_j \langle x_{i-k+1} x_{i-j+1}\rangle\right)$$

- divide through by $(N-1)$, and use $c_{-l} = c_l$,

$$c_k = \sum_{j=1}^{p} \phi_j c_{j-k}$$

- divide through by c_o,

$$r_k = \sum_{j=1}^{p} \phi_j r_{j-k}.$$

2.4 Lag p

- multiply by x_{i-p-1},

$$x_{i-p+1}x_{i+1} = \sum_{j=1}^{p} \left(\phi_j x_{i-p+1}x_{i-j+1}\right) + x_{i-p+1}\xi_{i+1},$$

- take expectance,

$$\langle x_{i-p+1}x_{i+1}\rangle = \sum_{j=1}^{p} \left(\phi_j \langle x_{i-p+1}x_{i-j+1}\rangle\right) + \langle x_{i-p+1}\xi_{i+1}\rangle$$

- eliminate the zero correlation forcing term

$$\langle x_{i-p+1}x_{i+1}\rangle = \sum_{j=1}^{p} \left(\phi_j \langle x_{i-p+1}x_{i-j+1}\rangle\right)$$

- divide through by $(N-1)$, and use $c_{-l} = c_l$,

$$c_p = \sum_{j=1}^{p} \phi_j c_{j-p}$$

- divide through by c_o,

$$r_p = \sum_{j=1}^{p} \phi_j r_{j-p}.$$

2.5 Putting it All Together

Rewriting all the equations together yields

$$\begin{array}{ccccccccccccc}
r_1 & = & \phi_1 r_o & + & \phi_2 r_1 & + & \phi_3 r_2 & + & \cdots & + & \phi_{p-1} r_{p-2} & + & \phi_p r_{p-1} \\
r_2 & = & \phi_1 r_1 & + & \phi_2 r_o & + & \phi_3 r_1 & + & \cdots & + & \phi_{p-1} r_{p-3} & + & \phi_p r_{p-2} \\
 & & & & & & \vdots & & & & & & \\
r_{p-1} & = & \phi_1 r_{p-2} & + & \phi_2 r_{p-3} & + & \phi_3 r_{p-4} & + & \cdots & + & \phi_{p-1} r_o & + & \phi_p r_1 \\
r_p & = & \phi_1 r_{p-1} & + & \phi_2 r_{p-2} & + & \phi_3 r_{p-3} & + & \cdots & + & \phi_{p-1} r_1 & + & \phi_p r_o
\end{array}$$

which can also be written as

$$\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{p-1} \\ r_p \end{pmatrix} = \begin{pmatrix} r_o & r_1 & r_2 & \cdots & r_{p-2} & r_{p-1} \\ r_1 & r_o & r_1 & \cdots & r_{p-3} & r_{p-2} \\ & \vdots & & & \vdots & \\ r_{p-2} & r_{p-3} & r_{p-4} & \cdots & r_o & r_1 \\ r_{p-1} & r_{p-2} & r_{p-3} & \cdots & r_1 & r_o \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{pmatrix}.$$

Recalling that $r_o = 1$, the above equation is also

$$\underbrace{\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{p-1} \\ r_p \end{pmatrix}}_{\mathbf{r}} = \underbrace{\begin{pmatrix} 1 & r_1 & r_2 & \cdots & r_{p-2} & r_{p-1} \\ r_1 & 1 & r_1 & \cdots & r_{p-3} & r_{p-2} \\ & \vdots & & & \vdots & \\ r_{p-2} & r_{p-3} & r_{p-4} & \cdots & 1 & r_1 \\ r_{p-1} & r_{p-2} & r_{p-3} & \cdots & r_1 & 1 \end{pmatrix}}_{\mathbf{R}} \underbrace{\begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{pmatrix}}_{\boldsymbol{\Phi}}$$

or succinctly

$$\mathbf{R}\boldsymbol{\Phi} = \mathbf{r}. \tag{2}$$

Note that this is a well-posed system (with a square coefficients matrix $\mathbf{R}$), i.e., with the same number of constraints (equations, $\mathbf{R}$'s rows) as unknowns (the elements ϕ_j of the unknown vector $\boldsymbol{\Phi}$). Further, $\mathbf{R}$ is full-rank and symmetric, so that invertability is guaranteed,

$$\hat{\boldsymbol{\Phi}} = \mathbf{R}^{-1}\mathbf{r}.$$

3 The Yule-Walker Equations and the Partial Autocorrelation Function

Equation 2 provides a convenient recursion for computing the pacf. The first step is to compute the acf up to a reasonable cutoff, say $p \simeq N/4$. Next, let $\mathbf{r}^{(i)}$ denote

Equation 2's rhs for the $p = i$ case. Similarly, let $\mathbf{R}^{(i)}$ denote the coefficient matrix for the same case. Then

- loop on i, $1 \leq i \leq p$
  - compute $\mathbf{R}^{(i)}$ and $\mathbf{r}^{(i)}$
  - invert for $\hat{\mathbf{\Phi}}^{(i)}$,

$$\hat{\mathbf{\Phi}}^{(i)} = \left(\mathbf{R}^{(i)}\right)^{-1} \mathbf{r}^{(i)} = \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \vdots \\ \hat{\phi}_i \end{pmatrix}$$

  - discard all $\hat{\phi}_j$ for $1 \leq j \leq i - 1$
  - retain $\hat{\phi}_i$,

$$\text{pacf}(i) = \hat{\phi}_i$$

- end loop on i
- plot pacf(i) as a function of i.